

Classical Versions of q -Gaussian Processes: Conditional Moments and Bell's Inequality

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Abstract: We show that classical processes corresponding to operators which satisfy a q -commutative relation have linear regressions and quadratic conditional variances. From this we deduce that Bell's inequality for their covariances can be extended from $q = -1$ to the entire range $-1 \leq q < 1$.

Corrections of Sunday, February 29, 2004 at 15:40

The following corrections were found after the printed version appeared in Comm. Math. Phys., 219 (2001), pp. 259-270.

1. Of course, the expected value $\mathbb{E} : \mathcal{A} \rightarrow \mathbb{C}$, not into \mathbb{R} .
2. Conditional variance in Proposition 2 is now correct.

1. Introduction

In this paper we consider a linear mapping $\mathcal{H} \ni f \mapsto \mathbf{a}_f \in \mathcal{B}$ from the real Hilbert space \mathcal{H} into the algebra \mathcal{B} of bounded operators acting on a complex Hilbert space which satisfies the q -commutation relations

$$\mathbf{a}_f \mathbf{a}_g^* - q \mathbf{a}_g^* \mathbf{a}_f = \langle f, g \rangle \mathbf{I}, \quad (1)$$

and $\mathbf{a}_f \Phi = 0$ for a vacuum vector Φ . This defines a non-commutative stochastic process $\mathbf{X}_f = \mathbf{a}_f + \mathbf{a}_f^*$, first studied in [5], which following [2] we call the q -Gaussian process. For different values of q , these processes interpolate between the bosonic ($q = 1$) and fermionic ($q = -1$) processes, and include free processes of Voiculescu [7] ($q = 0$).

One of the basic problems arising in this context is the existence of the classical versions of q -Gaussian processes, see Definition 2. For $q = 1$, these are

the classical Gaussian processes with the covariances $\langle f, g \rangle_{f, g \in \mathcal{H}}$. For $q = -1$, the classical versions are two-valued, so Bell's inequality [1] shows that only some covariances may have the classical versions. In [5] classical versions were constructed for covariances corresponding to stationary two-valued Markov processes ($q = -1$). In [2, Prop. 3.9], the existence of such classical versions was proved for all $-1 < q < 1$ in the case where the q -Gaussian process is Markovian (which can be characterized in terms of the covariance function).

The situation for other covariances remained open in [2] and it was unclear which q -Gaussian processes have no classical realizations. This issue is addressed in the present paper. Using a formula for conditional variances of classical versions we derive a constraint on the covariance which extends one of the Bell's inequalities from $q = -1$ to general $-1 \leq q < 1$. The inequality implies that there are covariances such that the corresponding non-commutative q -Gaussian processes cannot have classical versions over the entire range $-1 \leq q < 1$. Since q interpolates between the values $q = -1$, where classical versions may fail to exist and $q = 1$, where the classical versions always exist, it is interesting that there is a version of Bell's inequality which does not depend on q .

The proof relies on formulas for conditional moments of the first two orders, which are of independent interest. Computations to derive them were possible thanks to recent advances in the Fock space representation of q -commutation relations (1), see [2, 3].

2. Preliminaries

This section introduces the Fock space representation of q -Gaussian processes, and states known results in the form convenient for us. It is based on [2].

2.1. Notation. Throughout the paper, q is a fixed parameter and $-1 < q < 1$. For $n = 0, 1, 2, \dots$ we define q -integers $[n]_q := \frac{1-q^n}{1-q}$. The q -factorials are $[n]_q! := [1]_q[2]_q \dots [n]_q$, with the convention $[0]_q! := 1$.

The q -Hermite polynomials are defined by the recurrence

$$xH_n(x) = H_{n+1}(x) + [n]_q H_{n-1}(x), \quad n \geq 0 \quad (2)$$

with $H_{-1}(x) := 0, H_0(x) := 1$. These polynomials are orthogonal with respect to the unique absolutely continuous probability measure $\nu_q(dx) = f_q(x)dx$ supported on $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$, where density $f_q(x)$ has explicit product expansion, see [2, Theorem 1.10] or [6]; the second moments of q -Hermite polynomials are $\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} (H_n(x))^2 \nu_q(dx) = [n]_q!$. In our notation we are suppressing the dependence of $H_n(x)$ on q .

2.2. q -Fock space. For a real Hilbert space \mathcal{H} with complexification $\mathcal{H}_c = \mathcal{H} \oplus i\mathcal{H}$ we define its q -Fock space $\Gamma_q(\mathcal{H})$ as the closure of $\mathbb{C}\Phi \oplus \bigoplus_n \mathcal{H}_c^{\otimes n}$, the linear span of vectors $f_1 \otimes \dots \otimes f_n$, in the scalar product

$$\langle f_1 \otimes \dots \otimes f_n | g_1 \otimes \dots \otimes g_m \rangle_q = \begin{cases} \sum_{\sigma \in S_n} q^{|\sigma|} \prod_{j=1}^n \langle f_j, g_{\sigma(j)} \rangle & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \quad (3)$$

Here Φ is the vacuum vector, S_n are permutations of $\{1, \dots, n\}$ and $|\sigma| = \#\{(i, j) : i < j, \sigma(i) > \sigma(j)\}$. For the proof that (3) indeed is non-negative definite, see [3].

Given the q -Fock space $\Gamma_q(\mathcal{H})$ and $f \in \mathcal{H}$ we define the creation operator $\mathbf{a}_f : \Gamma_q(\mathcal{H}) \rightarrow \Gamma_q(\mathcal{H})$ and its $\langle \cdot | \cdot \rangle_q$ -adjoint, the annihilation operator $\mathbf{a}_f^* : \Gamma_q(\mathcal{H}) \rightarrow \Gamma_q(\mathcal{H})$ as follows:

$$\mathbf{a}_f \Phi := 0,$$

$$\mathbf{a}_f f_1 \otimes \dots \otimes f_n := \sum_{j=1}^n q^{j-1} \langle f, f_j \rangle f_1 \otimes \dots \otimes f_{j-1} \otimes f_{j+1} \otimes \dots \otimes f_n, \quad (4)$$

and

$$\begin{aligned} \mathbf{a}_f^* \Phi &= f, \\ \mathbf{a}_f^* f_1 \otimes \dots \otimes f_n &:= f \otimes f_1 \otimes \dots \otimes f_n. \end{aligned} \quad (5)$$

These operators are bounded, satisfy commutation relation (1), and $\mathbf{a}_{f+g} = \mathbf{a}_f + \mathbf{a}_g$, see [3].

2.3. q -Gaussian processes. We now consider (non-commutative) random variables as the elements of the algebra \mathcal{A} generated by the self-adjoint operators $\mathbf{X}_f := \mathbf{a}_f + \mathbf{a}_f^*$, with vacuum expectation state $\mathbb{E} : \mathcal{A} \rightarrow \mathbb{C}$ given by $\mathbb{E}(\mathbf{X}) = \langle \Phi | \mathbf{X} \Phi \rangle_q$.

Definition 1. We will call $\{\mathbf{X}(t) : t \in T\}$ a q -Gaussian (non-commutative) process indexed by T if there are vectors $h(t) \in \mathcal{H}$ such that $\mathbf{X}(t) = \mathbf{X}_{h(t)}$.

For a q -Gaussian process the covariance function $c_{t,s} := \mathbb{E}(\mathbf{X}_t \mathbf{X}_s)$ becomes $c_{t,s} = \langle h(t), h(s) \rangle$.

The Wick products $\psi(f_1 \otimes \dots \otimes f_n) \in \mathcal{A}$ are defined recurrently by $\psi(\Phi) := \mathbf{I}$, and

$$\begin{aligned} \psi(f \otimes f_1 \otimes \dots \otimes f_n) &:= \\ \mathbf{X}_f \psi(f_1 \otimes \dots \otimes f_n) - \sum_{j=1}^n q^{j-1} \langle f, f_j \rangle \psi(f_1 \otimes \dots \otimes f_{j-1} \otimes f_{j+1} \otimes \dots \otimes f_n). \end{aligned} \quad (6)$$

An important property of Wick products is that if $\mathbf{X} = \psi(f_1 \otimes \dots \otimes f_n)$ then

$$\mathbf{X} \Phi = f_1 \otimes \dots \otimes f_n. \quad (7)$$

We will also use the connection with q -Hermite polynomials. If $\|f\| = 1$ then

$$\psi(f^{\otimes n}) = H_n(\mathbf{X}_f), \quad (8)$$

see [2, Prop. 2.9]. Formulas (3), (7), and (8) show that for a unit vector $f \in \mathcal{H}$ we have

$$\mathbb{E} \left((H_n(\mathbf{X}_f))^2 \right) = \sum_{\sigma \in S_n} q^{|\sigma|} = [n]_q!. \quad (9)$$

Thus ν_q is indeed the distribution of \mathbf{X}_f . Our main use of the Wick product is to compute certain conditional expectations.

2.4. Conditional expectations. Recall that a (non-commutative) conditional expectation on the probability space $(\mathcal{A}, \mathbb{E})$ with respect to the subalgebra $\mathcal{B} \subset \mathcal{A}$ is a mapping $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$\mathbb{E}(\mathbf{Y}_1 \mathbf{X} \mathbf{Y}_2) = \mathbb{E}(\mathbf{Y}_1 \mathcal{E}(\mathbf{X}) \mathbf{Y}_2) \quad (10)$$

for all $\mathbf{X} \in \mathcal{A}, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{B}$.

We will study only algebras \mathcal{B} generated by the identity and the finite number of random variables $\mathbf{X}_{f_1}, \dots, \mathbf{X}_{f_n}$. In this situation, we will use a more probabilistic notation:

$$\mathbb{E}(\mathbf{X} | \mathbf{X}_{f_1}, \dots, \mathbf{X}_{f_n}) := \mathcal{E}(\mathbf{X}), \mathbf{X} \in \mathcal{A}.$$

In this setting conditional expectations are easily computed for \mathbf{X} given by Wick products. This important result comes from [2, Theorem 2.13].

Theorem 1. *If $\mathbf{Y} = \psi(g_1 \otimes \dots \otimes g_m)$, $\mathbf{X}_1 = \mathbf{X}_{f_1}, \dots, \mathbf{X}_k = \mathbf{X}_{f_k}$ for some $f_i, g_j \in \mathcal{H}$ and $P : \mathcal{H} \rightarrow \mathcal{H}$ denotes orthogonal projection onto the span of f_1, \dots, f_k then*

$$\mathbb{E}(\mathbf{Y} | \mathbf{X}_1, \dots, \mathbf{X}_k) = \psi(Pg_1 \otimes \dots \otimes Pg_m).$$

The following formula is an immediate consequence of Theorem 1 and (8), and is implicit in [2, Proof of Theorem 4.6].

Corollary 1. *If $\mathbf{X} = \mathbf{X}_f, \mathbf{Y} = \mathbf{X}_g$ with unit vectors $\|f\| = \|g\| = 1$ and H_n is the n^{th} q -Hermite polynomial, see (2), then*

$$\mathbb{E}(H_n(\mathbf{Y}) | \mathbf{X}) = \langle f, g \rangle^n H_n(\mathbf{X}). \quad (11)$$

For a finite number of vectors $f_0, f_1, \dots, f_k \in \mathcal{H}$, let $\mathbf{X}_k := \mathbf{X}_{f_k}$. These (non-commutative) random variables have linear regressions and constant conditional variances like the classical (commutative) Gaussian random variables.

Proposition 1.

$$\mathbb{E}(\mathbf{X}_0 | \mathbf{X}_1, \dots, \mathbf{X}_k) = \sum_{j=1}^k a_j \mathbf{X}_j \quad (12)$$

and

$$\mathbb{E}(\mathbf{X}_0^2 | \mathbf{X}_1, \dots, \mathbf{X}_k) = \left(\sum_{j=1}^k a_j \mathbf{X}_j \right)^2 + c \mathbf{I}. \quad (13)$$

If $f_1, \dots, f_k \in \mathcal{H}$ are linearly independent then the coefficients a_j, c are uniquely determined by the covariance matrix $C = [c_{i,j}] := [\langle f_i, f_j \rangle]$.

Notice that Eq. (13) can indeed be rewritten as the statement that conditional variance is constant,

$$\text{Var}(\mathbf{X}_0 | \mathbf{X}_1, \dots, \mathbf{X}_k) := \mathbb{E} \left((\mathbf{X}_0 - \mathbb{E}(\mathbf{X}_0 | \mathbf{X}_1, \dots, \mathbf{X}_k))^2 | \mathbf{X}_1, \dots, \mathbf{X}_k \right) = c \mathbf{I}.$$

Proof. This follows from Theorem 1 and (6). Write the orthogonal projection of f_0 onto the span of f_1, \dots, f_k as the linear combination $g = \sum_j a_j f_j$. Then $\mathbb{E}(\mathbf{X}_0 | \mathbf{X}_1, \dots, \mathbf{X}_k) = E(\psi(f_0) | \mathbf{X}_1, \dots, \mathbf{X}_k) = \psi(g) = \sum_j a_j \psi(f_j)$, which proves (12). Similarly, $\mathbb{E}(\mathbf{X}_0^2 - \|f_0\|^2 \mathbf{I} | \mathbf{X}_1, \dots, \mathbf{X}_k) = \mathbb{E}(\psi(f_0 \otimes f_0) | \mathbf{X}_1, \dots, \mathbf{X}_k) = \psi(g \otimes g) = (\sum_j a_j \mathbf{X}_j)^2 - \|g\|^2 \mathbf{I}$. This proves (13) with $c = \|f_0\|^2 - \|g\|^2$.

If $f_1, \dots, f_k \in \mathcal{H}$ are linearly independent then the representation $g = \sum_j a_j f_j$ is unique.

To analyze standardized triplets in more detail we need the explicit form of the coefficients. (We omit the straightforward calculation.)

Corollary 2. *If $\mathbf{X} := \mathbf{X}_f, \mathbf{Y} := \mathbf{X}_g, \mathbf{Z} := \mathbf{X}_h$ and $f, h \in \mathcal{H}$ are linearly independent unit vectors, then*

$$\mathbb{E}(\mathbf{Y} | \mathbf{X}, \mathbf{Z}) = a\mathbf{X} + b\mathbf{Z}, \quad (14)$$

$$\mathbb{E}(\mathbf{Y}^2 | \mathbf{X}, \mathbf{Z}) = (a\mathbf{X} + b\mathbf{Z})^2 + c\mathbf{I}, \quad (15)$$

where

$$a = \frac{\langle f, g \rangle - \langle g, h \rangle \langle f, h \rangle}{1 - \langle f, h \rangle^2}, \quad (16)$$

$$b = \frac{\langle g, h \rangle - \langle f, g \rangle \langle f, h \rangle}{1 - \langle f, h \rangle^2}. \quad (17)$$

Another calculation shows that $c = \det(\mathbf{C}) / (1 - \langle f, h \rangle^2)$, where \mathbf{C} is the covariance matrix; in particular $c \geq 0$.

3. Conditional Moments of Classical Versions

We give the definition of a classical version which is convenient for bounded processes; for a more general definition, see [2, Def. 3.1].

Definition 2. *A classical version of the process $\mathbf{X}(t)$ indexed by $t \in T \subset \mathbb{R}$ is a stochastic process $\tilde{\mathbf{X}}(t)$ defined on some classical probability space such that for any finite number of indexes $t_1 < t_2 < \dots < t_k$ and any polynomials P_1, \dots, P_k ,*

$$\begin{aligned} \mathbb{E}(P_1(\mathbf{X}(t_1))P_2(\mathbf{X}(t_2)) \dots P_k(\mathbf{X}(t_k))) = \\ E\left(P_1(\tilde{\mathbf{X}}(t_1))P_2(\tilde{\mathbf{X}}(t_2)) \dots P_k(\tilde{\mathbf{X}}(t_k))\right). \end{aligned} \quad (18)$$

Here $E(\cdot)$ denotes the classical expected value given by Lebesgue integral with respect to the classical probability measure.

Our main interest is in finite index set $T = \{t_1, t_2, t_3\}$, where $t_1 < t_2 < t_3$. In this case we write $\mathbf{X} := \mathbf{X}(t_1), \mathbf{Y} := \mathbf{X}(t_2), \mathbf{Z} := \mathbf{X}(t_3)$. We say that an ordered triplet $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ has a classical version $\tilde{X}, \tilde{Y}, \tilde{Z}$, if $\mathbb{E}(P_1(\mathbf{X})P_2(\mathbf{Y})P_3(\mathbf{Z})) = E(P_1(\tilde{X})P_2(\tilde{Y})P_3(\tilde{Z}))$ for all polynomials P_1, P_2, P_3 .

The classical version of a non-commutative process is order-dependent, since the left-hand side of (18) may depend on the ordering of the variables, while the right-hand side does not. For specific example in the context of q -Gaussian random variables, see [5, formulas (2.64) and (2.65)].

3.1. Triplets. All pairs $(\mathbf{X}_f, \mathbf{X}_g)$ of q -Gaussian random variables have classical versions because $\mathbb{E}(\mathbf{X}_f^m \mathbf{X}_g^n) = \mathbb{E}(\mathbf{X}_f^m \mathbf{X}_g^n)$ for all integer m, n ; however, the classical version of a triplet may fail to exist. With this in mind we consider q -Gaussian triplets

$$\mathbf{X} := \mathbf{X}_f, \mathbf{Y} := \mathbf{X}_g, \mathbf{Z} := \mathbf{X}_h. \quad (19)$$

To simplify the notation we take unit vectors $\|f\| = \|g\| = \|h\| = 1$. We assume that there is a classical version $(\tilde{X}, \tilde{Y}, \tilde{Z})$ of $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, in this order.

From Corollary 2 we know that non-commutative random variables $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ have linear regression and constant conditional variance. It turns out that the corresponding classical random variables $\tilde{X}, \tilde{Y}, \tilde{Z}$ also have linear regressions, while their conditional variances get perturbed into quadratic polynomials.

Theorem 2. *If $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is a classical version of the q -Gaussian triplet (19) then*

$$E(\tilde{Y}|\tilde{X}, \tilde{Z}) = a\tilde{X} + b\tilde{Z}, \quad (20)$$

$$E(\tilde{Y}^2|\tilde{X}, \tilde{Z}) = A\tilde{X}^2 + B\tilde{X}\tilde{Z} + C\tilde{Z}^2 + D, \quad (21)$$

where a, b are given by (16), (17),

$$A = \frac{ab(1-q)\langle f, h \rangle + a^2(1-q\langle f, h \rangle^2)}{1-q\langle f, h \rangle^2}, \quad (22)$$

$$B = \frac{ab(1+q)(1-\langle f, h \rangle^2)}{1-q\langle f, h \rangle^2}, \quad (23)$$

$$C = \frac{ab(1-q)\langle f, h \rangle + b^2(1-q\langle f, h \rangle^2)}{1-q\langle f, h \rangle^2}, \quad (24)$$

and

$$D = 1 - A - B\langle f, h \rangle - C. \quad (25)$$

The proof relies on the following technical result.

Lemma 1. *If H_n, H_m are q -Hermite polynomials given by (2), then*

$$\mathbb{E}(H_n(\mathbf{X})\mathbf{Z}\mathbf{X}H_m(\mathbf{Z})) = \begin{cases} \langle f, h \rangle^{n+1} [n+2]_q! & \text{if } m = n+2 \\ \langle f, h \rangle^{n-1} [n]_q! & \text{if } m = n-2 \\ \langle f, h \rangle^{n-1} (([n]_q + 1)\langle f, h \rangle^2 + q[n]_q) [n]_q! & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}, \quad (26)$$

$$\mathbb{E}(H_n(\mathbf{X})\mathbf{X}\mathbf{Z}H_m(\mathbf{Z})) = \begin{cases} \langle f, h \rangle^{n+1} [n+2]_q! & \text{if } m = n+2 \\ \langle f, h \rangle^{n-1} [n]_q! & \text{if } m = n-2 \\ \langle f, h \rangle^{n-1} ([n+1]_q \langle f, h \rangle^2 + [n]_q) [n]_q! & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}, \quad (27)$$

$$\mathbb{E}(H_n(\mathbf{X})\mathbf{X}^2 H_m(\mathbf{Z})) = \mathbb{E}(H_m(\mathbf{X})\mathbf{Z}^2 H_n(\mathbf{Z})) = \begin{cases} \langle f, h \rangle^{n+2} [n+2]_q! & \text{if } m = n+2 \\ \langle f, h \rangle^{n-2} [n]_q! & \text{if } m = n-2 \\ \langle f, h \rangle^n ([n+1]_q + [n]_q) [n]_q! & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}, \quad (28)$$

$$\mathbb{E}(H_n(\mathbf{X})H_m(\mathbf{Z})) = \begin{cases} \langle f, h \rangle^n [n]_q! & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}. \quad (29)$$

Proof of Theorem 2. Since $\tilde{X}, \tilde{Y}, \tilde{Z}$ are bounded random variables, to prove (20) we need only to verify that for arbitrary polynomials P, Q ,

$$E\left(P(\tilde{X})\tilde{Y}Q(\tilde{Z})\right) = E\left(P(\tilde{X})(a\tilde{X} + b\tilde{Z})Q(\tilde{Z})\right).$$

This is equivalent to

$$\mathbb{E}(P(\mathbf{X})\mathbf{Y}Q(\mathbf{Z})) = \mathbb{E}(P(\mathbf{X})(a\mathbf{X} + b\mathbf{Z})Q(\mathbf{Z})),$$

see (18). The latter follows from (14) and (10), proving (20).

To prove (21), we verify that for arbitrary polynomials P, Q we have

$$E\left(P(\tilde{X})\tilde{Y}^2Q(\tilde{Z})\right) = E\left(P(\tilde{X})(A\tilde{X}^2 + B\tilde{X}\tilde{Z} + C\tilde{Z}^2 + D)Q(\tilde{Z})\right).$$

By definition (18), this is equivalent to

$$\mathbb{E}(P(\mathbf{X})\mathbf{Y}^2Q(\mathbf{Z})) = \mathbb{E}(P(\mathbf{X})(A\mathbf{X}^2 + B\mathbf{X}\mathbf{Z} + C\mathbf{Z}^2 + D)Q(\mathbf{Z})). \quad (30)$$

It suffices to show that (30) holds true when $P = H_n$ and $Q = H_m$ are the q -Hermite polynomials defined by (2). Formula (15) implies that the left-hand side of (30) is given by

$$\begin{aligned} c\mathbb{E}(H_n(\mathbf{X})H_m(\mathbf{Z})) + a^2\mathbb{E}(H_n(\mathbf{X})\mathbf{X}^2H_m(\mathbf{Z})) + b^2\mathbb{E}(H_n(\mathbf{X})\mathbf{Z}^2H_m(\mathbf{Z})) \\ + ab\mathbb{E}(H_n(\mathbf{X})\mathbf{X}\mathbf{Z}H_m(\mathbf{Z})) + ab\mathbb{E}(H_n(\mathbf{X})\mathbf{Z}\mathbf{X}H_m(\mathbf{Z})), \end{aligned}$$

and the right-hand side becomes

$$\begin{aligned} A\mathbb{E}(H_n(\mathbf{X})\mathbf{X}^2H_m(\mathbf{Z})) + C\mathbb{E}(H_n(\mathbf{X})\mathbf{Z}^2H_m(\mathbf{Z})) + \\ B\mathbb{E}(H_n(\mathbf{X})\mathbf{X}\mathbf{Z}H_m(\mathbf{Z})) + D\mathbb{E}(H_n(\mathbf{X})H_m(\mathbf{Z})). \end{aligned}$$

Using formulas from Lemma 1 we can see that both sides are zero, except when $m = n$ or $m = n \pm 2$. We now consider these three cases separately.

Case $m = n + 2$: Using Lemma 1, (30) simplifies to

$$(a^2\langle f, h \rangle^2 + 2ab\langle f, h \rangle + b^2)\langle f, h \rangle^n [n+2]_q! = (A\langle f, h \rangle^2 + B\langle f, h \rangle + C)\langle f, h \rangle^n [n+2]_q!.$$

This equation is satisfied when coefficients A, B, C satisfy the equation

$$A\langle f, h \rangle^2 + B\langle f, h \rangle + C = a^2\langle f, h \rangle^2 + 2ab\langle f, h \rangle + b^2. \quad (31)$$

Case $m = n - 2$: Using Lemma 1, (30) simplifies to

$$(a^2 + 2ab\langle f, h \rangle + b^2\langle f, h \rangle^2)\langle f, h \rangle^{n-2} [n]_q! = (A + B\langle f, h \rangle + C\langle f, h \rangle^2)\langle f, h \rangle^{n-2} [n]_q!.$$

This equation is satisfied whenever

$$A + B\langle f, h \rangle + C\langle f, h \rangle^2 = a^2 + 2ab\langle f, h \rangle + b^2\langle f, h \rangle^2. \quad (32)$$

Case $m = n$: We use again Lemma 1. On both sides of Eq. (30) we factor out $\langle f, h \rangle^{n-1} [n]_q!$, and equate the remaining coefficients. (This is allowed since we are after sufficient conditions only!) We get

$$\begin{aligned} (\langle f, h \rangle (a^2 + b^2) ([n+1]_q + [n]_q) + ab (\langle f, h \rangle^2 [n+1]_q + (1+q) [n]_q) + c \langle f, h \rangle) = \\ ((1+q)(A+C) + B(q \langle f, h \rangle^2 + 1) [n]_q + D \langle f, h \rangle). \end{aligned}$$

Now we use $[n+1]_q = 1 + q[n]_q$. Suppressing the correction to the constant term (i.e., the term free of n), we get

$$\begin{aligned} (1+q) (\langle f, h \rangle (a^2 + b^2) + ab(1 + \langle f, h \rangle^2)) [n]_q + c \langle f, h \rangle + \dots = \\ ((1+q)(A+C) \langle f, h \rangle + B(q \langle f, h \rangle^2 + 1)) [n]_q + D \langle f, h \rangle, \end{aligned}$$

where $c + \dots$ denotes the suppressed constant term corrections.

This equation holds true when the coefficients at $[n]_q$ match, which gives

$$(1+q) \langle f, h \rangle (A+C) + B(q \langle f, h \rangle^2 + 1) = (1+q) ((a^2 + b^2) \langle f, h \rangle + ab(\langle f, h \rangle^2 + 1)), \quad (33)$$

and the constant terms match: $c + \dots = D$. The latter holds true when the expectations are equal ($n = m = 0$), and hence this condition is equivalent to (25). The remaining three equations (31), (32), and (33) have a unique solution given by the expressions (22), (23), (24). \square

Proof of Lemma 1. Using the definition of vacuum expectation state, (7) and (8) we get $\mathbb{E}(H_n(\mathbf{X}) \mathbf{Z} \mathbf{X} H_m(\mathbf{Z})) = \langle \mathbf{Z} H_n(\mathbf{X}) \Phi | \mathbf{X} H_m(\mathbf{Z}) \Phi \rangle_q = \langle \mathbf{X}_h f^{\otimes n} | \mathbf{X}_f h^{\otimes m} \rangle_q$. Therefore (4), and (5) imply

$$\begin{aligned} \mathbb{E}(H_n(\mathbf{X}) \mathbf{Z} \mathbf{X} H_m(\mathbf{Z})) = \\ \langle [n]_q \langle f, h \rangle f^{\otimes n-1} + h \otimes f^{\otimes n} | [m]_q \langle f, h \rangle h^{\otimes m-1} + f \otimes h^{\otimes m} \rangle_q. \end{aligned} \quad (34)$$

The latter is zero, except when $m = n$ or $m = n \pm 2$. We will consider these two cases separately.

If $m = n$, by orthogonality we have

$$\mathbb{E}(H_n(\mathbf{X}) \mathbf{Z} \mathbf{X} H_m(\mathbf{Z})) = [n]_q^2 \langle f, h \rangle^2 \langle f^{\otimes n-1} | h^{\otimes n-1} \rangle_q + \langle h \otimes f^{\otimes n} | f \otimes h^{\otimes n} \rangle_q.$$

Clearly, $\langle f^{\otimes n-1} | h^{\otimes n-1} \rangle_q = \langle f, h \rangle^{n-1} [n-1]_q!$; this can be seen either from (9) and (11), or directly from the definition (3).

By (3) the second term splits into the sum over permutations $\sigma' \in S_{n+1}$ such that $\sigma'(1) = 1$ and the sum over the permutations such that $\sigma'(1) = k > 1$. This gives

$$\begin{aligned} \langle h \otimes f^{\otimes n} | f \otimes h^{\otimes n} \rangle_q = \sum_{\sigma \in S_n} \langle f, h \rangle q^{|\sigma|} \langle f, h \rangle^n + \sum_{k=2}^{n+1} \sum_{\sigma \in S_n} q^{k-1+|\sigma|} \langle f, h \rangle^{n-1} = \\ \langle f, h \rangle^{n+1} [n]_q! + \langle f, h \rangle^{n-1} q [n]_q [n]_q!. \end{aligned}$$

Elementary algebra now yields (26) for $m = n$.

If $m = n + 2$, then the right-hand side of (34) consists of only one term we get

$$\begin{aligned}\mathbb{E}(H_n(\mathbf{X})\mathbf{Z}\mathbf{X}H_{n+2}(\mathbf{Z})) &= [n+2]_q \langle f, h \rangle \langle h \otimes f^{\otimes n} | h^{\otimes n+1} \rangle_q = \\ &= [n+2]_q \langle f, h \rangle \sum_{\sigma \in S_{n+1}} q^{|\sigma|} \langle f, h \rangle^n = \langle f, h \rangle^{n+1} [n+2]_q!.\end{aligned}$$

Since $m = n - 2$ is given by the same expression with the roles of m, n switched around, this ends the proof of (26).

The remaining expectations match the corresponding commutative values, and can also be evaluated using recurrence (2) and formulas (11), and (9).

To prove (27) notice that since \mathbf{X} and $H_n(\mathbf{X})$ commute, using (2) and (11) we get

$$\begin{aligned}\mathbb{E}(H_n(\mathbf{X})\mathbf{X}\mathbf{Z}H_m(\mathbf{Z})) &= \mathbb{E}(\mathbf{X}H_n(\mathbf{X})(H_{m+1}(\mathbf{Z}) + [m]_q H_{m-1}(\mathbf{Z}))) = \\ &= \langle f, h \rangle^{m+1} \mathbb{E}(\mathbf{X}H_n(\mathbf{X})H_{m+1}(\mathbf{X})) + [m]_q \langle f, h \rangle^{m-1} \mathbb{E}(\mathbf{X}H_n(\mathbf{X})H_{m-1}(\mathbf{X})).\end{aligned}$$

The only non-zero values are when $m = n$, or $m = n \pm 2$. Using (2) again, and then (9) we get (27).

Since by (11) we have

$$\mathbb{E}(H_n(\mathbf{X})\mathbf{X}^2 H_m(\mathbf{Z})) = \langle f, h \rangle^m \mathbb{E}\mathbf{X}^2 H_n(\mathbf{X})H_m(\mathbf{X}),$$

recurrence (2) used twice proves (28).

Formula (29) is an immediate consequence of (11) and (9). \square

3.2. Relation to processes with independent increments. In [2, Definition 3.5] the authors define the non-commutative q -Brownian motion and show that it has a classical version, see [2, Cor. 4.5]. Since the classical version of the q -Brownian motion is Markov, Theorem 2 implies that all regressions are linear, and all conditional variances are quadratic. A computation gives the following expression for the conditional variances.

Proposition 2. *Let \tilde{X}_t be the classical version of the q -Brownian motion, i. e., $\langle f_t, f_s \rangle = \min\{s, t\}$. Then for $t_1 < t_2 < \dots < t_n < s < t$ we have*

$$\begin{aligned}\text{Var}(\tilde{X}_s | \tilde{X}_{t_1}, \dots, \tilde{X}_{t_n}, X_t) &= \\ &= \frac{(t-s)(s-t_n)}{(t-qt_n)} \left(1 + \frac{(\tilde{X}_t - \tilde{X}_{t_n})(t\tilde{X}_t - t_n\tilde{X}_{t_n})(1-q)}{(t-t_n)^2} \right).\end{aligned}$$

In [8], classical processes with independent increments, linear regressions, and quadratic conditional variances are analyzed. These processes have the same covariances as q -Brownian motion, but the conditional variances are quadratic functions of the increment $\tilde{X}_t - \tilde{X}_{t_n}$ only. Proposition 2 shows that the classical realizations of q -Brownian motion are not among the processes in [8] and thus have dependent increments.

4. Bell's Inequality

It is well known that all q -Gaussian n -tuples with $q = 1$ have classical versions: these are given by the classical Gaussian distribution with the same covariance matrix $[\langle f_i, f_j \rangle]$.

For $q = -1$ the classical version of the the standardized q -Gaussian triplet $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ consists of the ± 1 -valued symmetric random variables. The celebrated Bell's inequality [1] therefore restricts their covariances:

$$1 - \langle f, h \rangle \geq |\langle f, g \rangle - \langle g, h \rangle|. \quad (35)$$

In particular, there are triplets of q -Gaussian random variables with $q = -1$ which do not have a classical version.

The following shows that restriction (35) is in force for sub-Markov covariances over the entire range $-1 \leq q < 1$.

Theorem 3. *Suppose that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is a classical version of q -Gaussian $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) := (\mathbf{X}_f, \mathbf{X}_g, \mathbf{X}_h)$, where $f, h \in \mathcal{H}$ are linearly independent, and $-1 \leq q < 1$. If either*

$$\langle f, g \rangle \langle g, h \rangle \leq \langle f, h \rangle \text{ and } 0 < \langle f, h \rangle < 1, \quad (36)$$

or $\langle f, h \rangle = 0$, or $q = -1$, then inequality (35) holds true.

Proof. Since the case $q = -1$ is well known, we restrict our attention to the case $-1 < q < 1$. Our starting point is expression (21). A computation shows that the conditional variance $Var(\tilde{Y}|\tilde{X}, \tilde{Z}) := E(\tilde{Y}^2|\tilde{X}, \tilde{Z}) - \left(E(\tilde{Y}|\tilde{X}, \tilde{Z})\right)^2$ is as follows.

$$Var(\tilde{Y}|\tilde{X}, \tilde{Z}) = 1 - a^2 - b^2 - ab\langle f, h \rangle \frac{(1+q)(1-\langle f, h \rangle^2) + 2(1-q)}{1-q\langle f, h \rangle^2} - \frac{ab(1-q)}{1-q\langle f, h \rangle^2} \left(\tilde{Z}\langle f, h \rangle - \tilde{X} \right) \left(\langle f, h \rangle \tilde{X} - \tilde{Z} \right). \quad (37)$$

The right-hand side of this expression must be non-negative over the support of \tilde{X}, \tilde{Z} . It is known, see [4, Lemma 8.1] or [2, Theorem 1.10], that \tilde{X}, \tilde{Z} have the joint probability density function $f(x, z)$ with respect to the product of marginals ν_q . Moreover, f is defined for all $-2/\sqrt{1-q} \leq x, z \leq 2/\sqrt{1-q}$ and from its explicit product expansion we can see that

$$f(x, z) \geq \prod_{k=0}^{\infty} \frac{(1 - \langle f, h \rangle^2 q^k)}{(1 + \langle f, h \rangle q^k)^4}$$

is strictly positive. In particular, the right-hand side of (37) must be non-negative when evaluated at $\tilde{X} = \sqrt{2}/\sqrt{1-q}, \tilde{Z} = -\sqrt{2}/\sqrt{1-q}$.

Using formulas (16), (17) with the above values of \tilde{X}, \tilde{Y} we get the rational expression for the conditional variance which can be written as follows.

$$\begin{aligned} (1 - q\langle f, h \rangle^2) (1 - \langle f, h \rangle)^2 Var(\tilde{Y}|\tilde{X}, \tilde{Z}) = \\ (1 - \langle f, h \rangle)^2 (1 - q\langle f, h \rangle^2 + (1 + q)\langle f, h \rangle \langle f, g \rangle \langle g, h \rangle) - \\ (\langle f, g \rangle - \langle g, h \rangle)^2 (1 + \langle f, h \rangle^2). \end{aligned} \quad (38)$$

Therefore

$$(1 - \langle f, h \rangle)^2 (1 - q\langle f, h \rangle^2 + (1 + q)\langle f, h \rangle \langle f, g \rangle \langle g, h \rangle) \geq (\langle f, g \rangle - \langle g, h \rangle)^2 (1 + \langle f, h \rangle^2).$$

Since the assumptions imply that $1 - q\langle f, h \rangle^2 + (1 + q)\langle f, h \rangle \langle f, g \rangle \langle g, h \rangle \leq 1 + \langle f, h \rangle^2$, this implies $(1 - \langle f, h \rangle)^2 \geq (\langle f, g \rangle - \langle g, h \rangle)^2$, proving (35).

4.1. Examples. The first example shows that there are covariances such that q -Gaussian random variables have no classical version for all $-1 \leq q < 1$.

Example 1. Consider the case $\langle f, h \rangle = \langle g, h \rangle > 0$. This can be realized when the covariance matrix is non-negative definite; a computation shows that this is equivalent to the condition $2\langle f, h \rangle^2 \leq 1 + \langle f, g \rangle$. Since (36) is satisfied, Bell's inequality (35) implies $1 + \langle f, g \rangle \geq 2\langle f, h \rangle$. Therefore, all choices of vectors $f, g, h \in \mathcal{H}$ such that $\langle f, h \rangle = \langle g, h \rangle$, $0 < \langle f, h \rangle < 1$, and $2\langle f, h \rangle^2 - 1 < \langle f, g \rangle < 2\langle f, h \rangle - 1$ lead to q -Gaussian triplets with no classical version for $-1 \leq q < 1$.

A nice feature of Theorem 3 is that its statement does not depend on q , as long as $q < 1$. But such a result cannot be sharp. A less transparent statement that the conditional variance is non-negative is a stronger restriction on the covariances, and it depends on q . This is illustrated by the next example.

Example 2. Suppose $\langle f, h \rangle = \langle g, h \rangle = 1/2$. Inequality (35) used in Example 1 implies that if a classical version of a q -Gaussian process exists then $\langle f, g \rangle \geq 0$. Evaluating the conditional variance $\text{Var}(\tilde{Y}|\tilde{X}, \tilde{Z})$ at $\tilde{X} = 2/\sqrt{1-q}$, $\tilde{Z} = -\tilde{X}$ we get a more restrictive constraint $\langle f, g \rangle \geq \frac{q+5}{36}$.

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